Truck Drivers, a Straw, and Two Glasses of Water

The Problem

It was a dark and stormy night. Unable to continue driving, two truck drivers, John and Bill, were stuck at the (clever name here) Truck Stop and decided to have dinner together. At the end of their meal, Bill noticed that his water glass was empty while John's glass was still full. Bill was terribly thirsty, and since the waitresses were extremely busy, he stuck his straw in John's glass, put his finger on the top to trap water, moved the straw over to his glass and released the water. He wondered what would happen if he did this over and over again. Would he ever end up with equal amounts of water in the two glasses?

Note: This problem was presented to the second author by an actual truck driver named John who assisted her with a move across the country. She sincerely hopes that he will enjoy this paper.

Can we answer Bill's question?

Mathematically, Bill's question is an interesting one and gives rise to several variations and generalizations of the problem, some of which we address in later sections. For the sake of simplicity, throughout the paper we will say that a *straw move* consists of putting a straw into John's glass of water until the straw touches the bottom. This ensures that the straw will be filled to the same height as the water in the glass. A finger is then placed on the top of the straw and the straw is moved to the second glass, where the water is released into the glass. See Figure 1 for a depiction of a straw move.

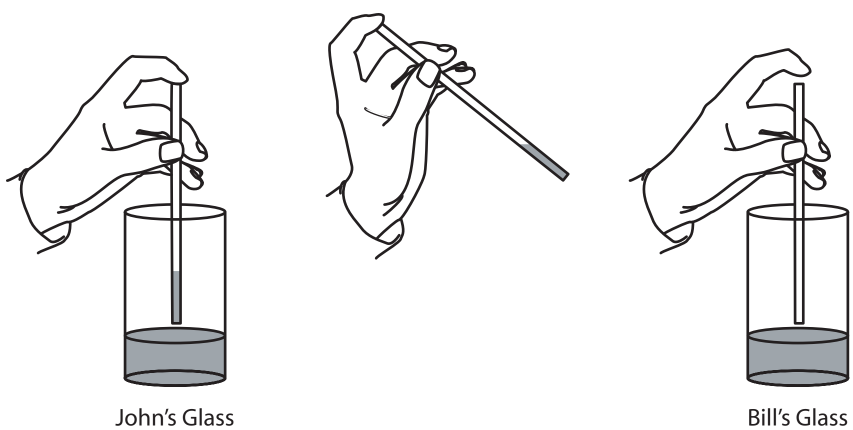

Figure 1: Transferring water with a "straw move"

We assume that the glasses are cylindrical in shape and are identical. The main consequence of this assumption is that the amount of water taken up by the straw during each straw move is a constant fraction of the amount of water in the glass and can be determined by computing the ratio of the volume of water in the straw to the volume of water in the glass. We will let p denote this ratio. If we let R be the radius of the glass, r be the radius of the straw and h be the height of the water, then

$$p = \frac{\pi r^2 h}{\pi R^2 h} = \frac{r^2}{R^2}.$$

Incidentally, after determining the relative size of a "standard" glass and a "standard" straw, the authors believe that a realistic value for p is .015.

In reality, we are not interested in the actual amount of water in both glasses in terms of ounces, but simply what fraction of the total water is in John's glass. In general, let x represent this fraction. Then $1-x$ represents the fraction of the water that is in Bill's glass.

In our truckstop scenario, since John's glass is initially full and Bill's glass is empty, $x = 1$. After one move John's glass contains $1-p$ of the total water and Bill's glass contains p of the water. For the second straw move, the straw will again contain p of the total water in John's glass, which is now equal to $(1-p)$, so the straw will remove $p(1-p)$ of the water. This leaves John's glass with $(1-p) - p(1-p) = (1-p)^2$ of the water and Bill's glass with $1-(1-p)+p(1-p) = 1-(1-p)^2$ of the water. Continuing in this manner, we can see that after k iterations of the straw move, John's glass will be left with $(1-p)^k$ of the total water and Bill's glass will contain $1-(1-p)^k$ of the water.

The question, then, is can John and Bill reach a point when the amount in the two glasses is equal, that is, when $(1-p)^k = 1-(1-p)^k$? This is the same as asking, can $(1-p)^k = \frac{1}{2}$? Solving this equation for p, we find that

$$1 - p = \sqrt[k]{\frac{1}{2}},$$

so

$$p = 1 - \frac{1}{\sqrt[k]{2}}.$$

Thus, we can only reach exactly half in each glass if the straw is the right size, namely $p = 1 - \frac{1}{\sqrt[k]{2}}$ for some value of k. If there is such a p value, then k is the number of straw moves it will take to reach equality.

For the empirically determined value of p as $p = .015$, it will take 46 straw moves to reach a distribution of water with 49.896% of the water in John's glass and 50.104% of the water in Bill's. Therefore, for this size straw, equality is never attained; one of the glasses always has more water than the other, but after 46 moves, it is very close (and likely indistinguishable to the human eye).

More general starting conditions

Suppose now that we do not necessarily start with John's glass full and Bill's empty, but simply with John's having more water than Bill's. Again we ask the question: can we reach exactly half of the water in each glass through a sequence of straw moves?

We again let x denote the fraction of the water that is in John's glass. This time, if the straw is put into John's glass it will pick up px of the water in the first straw move and transfer it to Bill's glass. Thus, after k moves, John's glass will be left with $(1-p)^k x$ of the water.

The question is thus reduced to determining the value of k for which

$$(1-p)^k x = \frac{1}{2}$$

Solving this equation gives

$$p = 1 - \frac{1}{\sqrt[k]{2x}}.$$

Thus, for a given starting amount of water x, equality can be reached if the fraction p of water in the straw is equal to $1 - \frac{1}{\sqrt[k]{2x}}$ for some k. If so, then k is the number of straw moves needed to reach equality.

Alternating Moves

Suppose we want to add a bit more flexibility in the movement of water between the glasses and we decide to allow straw moves both from John's glass to Bill's and from Bill's to John's. We call these *forward* and *reverse* moves. A forward move followed by a reverse move is called a *cycle* of moves. If we alternate between forward and reverse straw moves, can we reach a state with half of the water in each glass?

Assume that initially John's glass contains more water than Bill's, and let x be the fraction of water in John's glass so $x \geq \frac{1}{2}$. Define $f(x)$ to be the amount of water in John's glass after a forward straw move, so

$$f(x) = (1-p)x.$$

If we do a single reverse straw move, then John's glass will have $x + p(1-x) = (1-p)x + p$ amount of water in it. Let $g(x)$ be the amount of water in John's glass after an reverse straw move, so

$$g(x) = (1-p)x + p.$$

After a cycle of moves, the amount of water in John's glass is

$$g(f(x)) = g((1-p)x) = (1-p)(1-p)x + p = (1-p)^2 x + p.$$

Define $h(x) = g(f(x))$. One way to reach equality is after a reverse straw move, which means after k complete cycles of moves. The result of these k cycles can be determined by iterating the function $h(x)$ k times, that is, by determining $h^k(x)$. The other possibility is that equality is reached after a forward straw move, which means after k cycles of moves followed by one more forward straw move. The result is determined by iterating the $h(x)$ function k times and then computing $f(h^k(x))$.

Case 1 (An even number of moves): Can equality be reached after k complete cycles of moves? If so, then $h^k(x) = \frac{1}{2}$ for some k. A nice formula can be computed for the function $h^k(x)$, which is a geometric series.

$$\begin{aligned} h^k(x) &= (1-p)^{2k}x + p(1-p)^{2(k-1)} + p(1-p)^{2(k-2)} + \cdots + (1-p)^2 + p \\ &= (1-p)^{2k}x + p[(1-p)^{2(k-1)} + (1-p)^{2(k-2)} + \cdots + (1-p)^2 + 1] \\ &= (1-p)^{2k}x + p\left[\frac{1-(1-p)^{2k}}{1-(1-p)^2}\right] \end{aligned}$$

Note that the amount of water in John's glass is decreasing after each iteration of $h(x)$ since the amount of water being transferred in a reverse straw move is smaller than the amount of water being transferred in a forward straw move. Taking more and more iterations of $h(x)$ results in taking the limit as k goes to infinity of $h^k(x)$.

$$\begin{aligned}\lim_{k\to\infty} h^k(x) &= \lim_{k\to\infty} \frac{1-(1-p)^{2(k-1)}}{1-(1-p)^2} \\ &= \frac{1}{2-p}\end{aligned}$$

Thus the amount of water in John's glass is approaching $\frac{1}{2-p}$ and since $(2-p)$ is always less than 2, this amount of water is always greater than $\frac{1}{2}$, so equality is never reached after k complete cycles of moves.

Graphically, the iteration of the $h(x)$ function can be viewed in the following manner. Since the amount of water $h(x)$ is a fraction of the total amount of water, it will take on values between 0 and 1, so $h(x)$ can be graphed in the unit square. Note that $h(0) = p$, (which is small since we assume that the size of the radius of a straw relative to that of a glass is small) and $h(1) = (1-p)^2 + p$. The function $h(x) = (1-p)^2x + p$ is a linear function of x, and thus the graph is a line with slope $(1-p)^2$. Since x is the starting amount of water in John's glass, the y value $h(x)$ determines the amount of water in John's glass after one complete cycle of moves. The graph of $h(x)$ and the result of a forward straw move is shown in Figure 2 by the dotted lines.

For a second cycle of moves, $h(x)$ is our input value, which graphically is equivalent to turning the y value $h(x)$ into an x value using the line $y = x$, as shown in Figure 3.

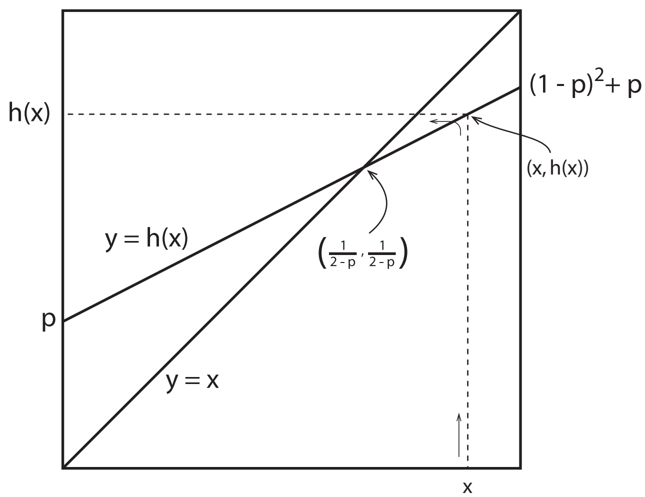

Figure 2: Using a graph to determine the result of a straw move.

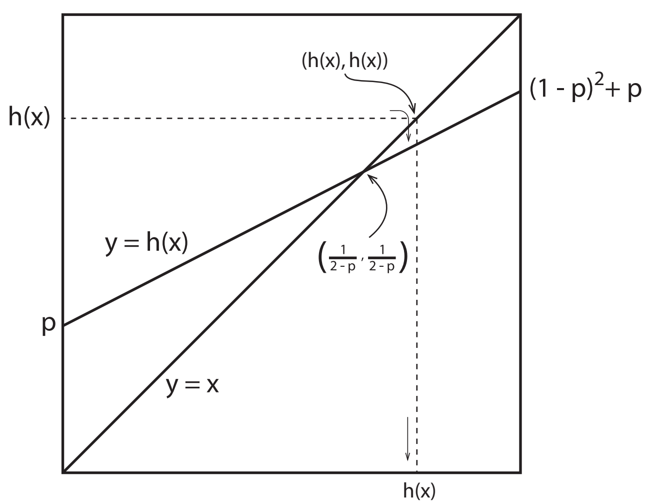

Figure 3: Turning $h(x)$ into an x value.

Figure 4 shows the combination of the ideas in Figure 2 and Figure 3 by starting with an initial fraction x of water in John's glass and showing the result of a complete cycle of moves.

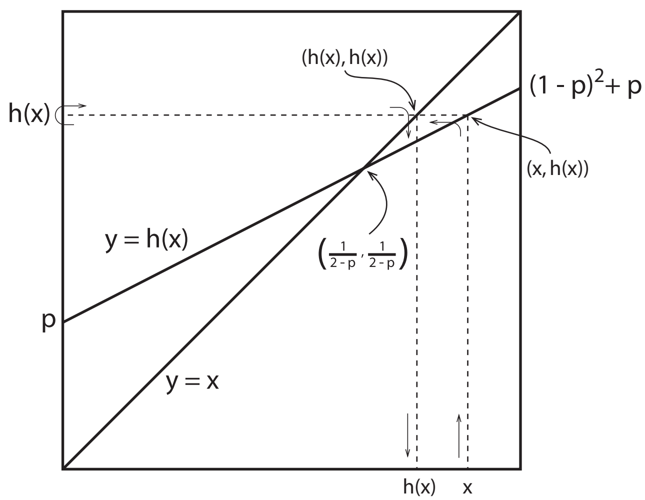

Figure 4: Determing the amount of water in John's glass after one complete cycle of moves.

Figure 5 shows that after iterating $h(x)$ multiple times, the amount of water in John's glass approaches the point of intersection of $h(x)$ and $y = x$, which is the point $(\frac{1}{(2-p)}, \frac{1}{(2-p)})$. Of course, the value $\frac{1}{(2-p)}$ is the value that was obtained algebraically by determining $\lim_{k\to\infty} h^k(x)$.

Case 2 (An odd number of moves): Can equality be reached after k complete cycles of moves followed by a single forward straw move? This would mean that $f(h^k(x)) = \frac{1}{2}$ for some value of k. Again this question will be answered both algebraically and graphically.

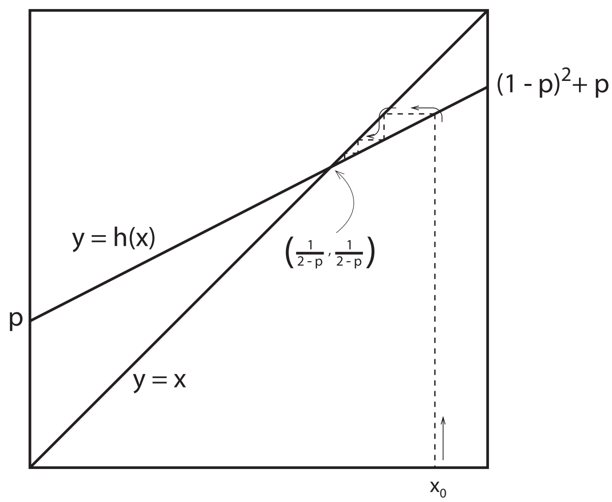

Figure 5: The amount of water in John's glass approaches a limiting value as multiple iterations of water movement are performed.

Our formula for $h^k(x)$ can be used to find that

$$f(h^k(x)) = (1-p)\left[(1-p)^{2k}x + p\frac{1-(1-p)^{2k}}{1-(1-p)^2}\right],$$

which converges to $\frac{1-p}{2-p} = 1 - \frac{1}{2-p}$.

Note that since $1-p$ is between 0 and 1, the sequence of x values after an odd number of moves is term by term less than the sequence of x values after an even number of moves. This is obvious since an odd number of moves ends with a transfer of water away from John's glass while an even number of moves ends with a transfer of water into John's glass. In addition, the sequence of x values corresponding to an even number of moves is a decreasing sequence converging to $\frac{1}{2-p}$ from above. Since $f(x)$ is continuous and monotonically increasing, this means that the sequence of x values after an odd number of moves is also a decreasing sequence, converging to $f(\frac{1}{2-p}) = 1 - \frac{1}{2-p}$. Since $\frac{1}{2-p}$ is between $\frac{1}{2}$ and 1 then $1 - \frac{1}{2-p}$ is between 0 and $\frac{1}{2}$. It is possible that along the way, the value $\frac{1}{2}$ is obtained, but generally the sequence may pass $\frac{1}{2}$ without hitting it exactly.

In summary, in the process of moving water back and forth between John and Bill's glasses, the amount of water in John's glass decreases, then increases, then decreases, and so on in such a way as to generally decrease. The amount in John's glass after odd iterations decreases to the value $1 - \frac{1}{2-p}$ which is less than $\frac{1}{2}$. The amount after even iterations decreases to $\frac{1}{2-p}$ which is greater than $\frac{1}{2}$. The odd iterations, then, are the only ones that might achieve the equality of $x = \frac{1}{2}$ that John was hoping for. At the very least, the amount in John's glass passes $x = \frac{1}{2}$ on its descent to $1 - \frac{1}{2-p} < \frac{1}{2}$.

The limiting situation is thus one in which the glasses oscillate between the two states: one in which John's glass has close to $\frac{1}{2-p}$ and one in which John's glass has close to $1 - \frac{1}{2-p}$. In dynamical systems [1] this is called a "limit cycle" because the values do not approach a single value but a cycle of two different limits. Figure 6 shows the graph of the limit cycle obtained by alternating between applying $f(x)$ and $g(x)$ to the amounts of water in John's glass, which is equivalent to alternating between forward and reverse straw moves.

Mathematically, there is a clear distinction between $x = \frac{1}{2}$ and $x = \frac{1}{2-p}$ but realistically would the difference between these values be discernible to the human eye? A few practical considerations have clearly been neglected. Water is made up of molecules so it is not possible to have a continuum of values for

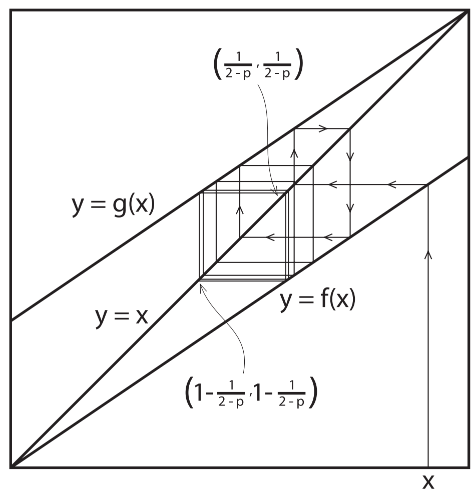

Figure 6: A graph of the ”limit cycle”.

x. Furthermore, water has surface tension, and this has not been analyzed. Although these effects are small, so is the precision of the limiting process. In light of this, can John and Bill's question really be answered using the above analysis? For the value $p = .015$ which was found to be a good approximation for a standard straw, the amount in Bill's glass ($\frac{1}{2-p}$) will be approximately .5038 which is indiscernible from $\frac{1}{2}$ by the human eye. Thus for all practical purposes, the limiting situation described above will appear to the human eye as having reached equality with both glasses containing $\frac{1}{2}$ of the water.

For what values do we reach equality?

We have observed that the value $x = \frac{1}{2}$ might be obtained after some number k of complete cycles of straw moves followed by a single forward straw move.

Under what circumstances will this happen? Intuitively, one might guess that this is a very rare situation, since iterating $h(x)$ over and over again gives a decreasing sequence of points between 0 and 1, but any given value in this interval is not very likely to be obtained.

One way to answer this question is to work backwards. Assuming an even split of water has been reached, i.e. $x = \frac{1}{2}$, what could the amounts of water in John and Bill's glasses have been one straw move prior to obtaining that state? What would the amount in the glasses be one complete cycle of moves prior to equality? If John's glass ever has $x = \frac{1}{2}$, then as in the previous section, it must be after an odd number of moves. Therefore, we can solve

$$f(x) = (1-p)x = \frac{1}{2}$$

to obtain $x = \frac{1}{2(1-p)}$.

Thus, if the original amount of water in John's glass is $\frac{1}{2(1-p)}$, equality will be reached after one forward straw move. Note that if $x < \frac{1}{2(1-p)} < \frac{1}{2}$ then equality will never be attained since the amount in John's glass decreases to $\frac{1}{2-p}$ and x is already less than $\frac{1}{2}$. If $x > \frac{1}{2(1-p)}$ then we want to determine if $x = \frac{1}{2}$ can be obtained after an odd number of moves, which is equivalent to obtaining $x = \frac{1}{2-p}$ after an even number of moves (i.e. after some number of iterations of $h(x)$). This means solving for x in the following equation:

$$h^k(x) = \frac{1}{2(1-p)}.$$

Since

$$h(x) = (1-p)^2 x + p,$$

the solution for $k = 1$ can be found as follows:

$$\begin{aligned}
(1-p)^2 x + p &= \frac{1}{2(1-p)} \\
x &= \frac{1}{2(1-p)^3} - \frac{p}{(1-p)^2}.
\end{aligned}$$

More generally, we can define a sequence with $a_1 = 1/2$, and

$$\begin{aligned}
h(a_n) &= a_{n-1} \\
(1-p)^2 a_n + p &= a_{n-1} \\
a_n &= \frac{a_{n-1} - p}{(1-p)^2}
\end{aligned}$$

This means that if at any point John's glass contains a_n amount of water, then equality will be reached after some number of iterations of $h(x)$, followed by a single application of $f(x)$. Note that going backward in this iteration is not the same as a straw move going in the other direction; in this case we are analyzing what happened prior to a straw move.

To graphically determine what possible initial amounts of water x and $1-x$ could lead to attaining half of the water in each glass, use the graphs of $f(x)$ and $h(x)$ (See Figure 7).

Start at the point $(0, 1/2)$ on the y-axis. Move horizontally to the graph of $y = f(x)$ and then vertically down to the x-axis to obtain the point $x_1 = \frac{1}{2(1-p)}$ (this represents the first initial state from which it is possible to obtain half of the water in each glass). Now move vertically from x_1 to the point $y = x = \frac{1}{2(1-p)}$

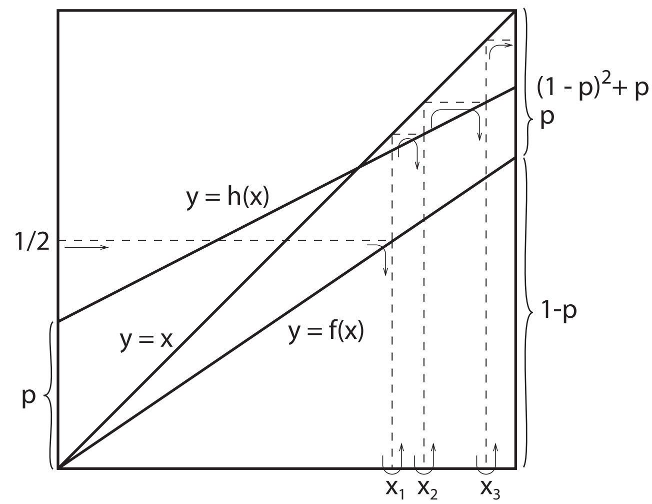

Figure 7: Possible initial water amounts that will result in an even split of water.

on the line $y = x$ and move horizontally to the graph of $h(x)$. Note that $1/(2(1-p)) > 1/(2-p)$, so this starting point is above the point of intersection between the graph of $h(x)$ and $y = x$. Note also that the graph of $h(x)$ takes values between $h(0) = p$ and $h(1) = 1 - p + p^2$. If x is smaller than p or bigger than $1 - p + p^2$, then this is impossible and one will never obtain the state with half of the water in each glass. Assuming that x is between those values, though, then it is possible to move horizontally down the graph of $h(x)$. From the graph of $h(x)$, move vertically up to the x-axis. This value of x represents a second initial amount of water from which it is possible to obtain half of the water in each glass. Iterate this procedure. Depending on the value of p, it will take a certain amount of time before $x < 0$ or $x > 1$. When this happens there are no further solutions possible (see Figure 7). Geometrically, it is apparent that for any value of p, only finitely many initial values of x that arise in this way.

As p gets closer to zero, the number of possible initial volumes increases and as p gets closer to $\frac{1}{2}$ the number of initial values decreases. If $p > \frac{1}{2}$, there are no possible initial volumes because then $1/(2(1-p)) > 1$.

What if John's glass is not as full as Bill's?

Where did we use the assumption that John's glass was fuller than Bill's? It was when we analyzed the sequence $x, h(x), h(h(x)), \ldots$ and saw that it approached the value $1/(2-p)$ if we assumed that $x > \frac{1}{2-p}$. If $x < 1/(2-p)$, the same analysis shows x will again approach the value $1/(2-p)$.

This case is simply the situation in which the roles of Bill's and John's glasses are reversed after the first move of water from John's glass to Bill's. Which starting positions of this type can lead to $x = 1/2$? Luckily, $x = 1/2$ is exactly the same under exchanging John's and Bill's glasses, so this case can be studied by looking at the finite sequence of starting positions for John's glass and stipulating that it is Bill who starts with that quantity of water instead of John.

Finally, what if the starting position is $x = \frac{1}{2-p}$? Then one move brings the amount to $1 - \frac{1}{2-p}$ and the next move returns it to $\frac{1}{2-p}$.

An elegant example

Theoretical mathematicians may find it interesting not to talk about straws in real life but rather to examine special cases that are mathematically beautiful. In the special case that $x_0 = 1$ and $1 - x_0 = 0$ and $p = \frac{1}{2}$ (This is a rather large straw!) an interesting pattern emerges by alternating the movement of water between the two glasses. If the volume of water is expressed in binary form, we obtain the following:

	John's Glass	Bill's Glass
Starting position	1.000000 . . .	0.000000 . . .
Iteration 1	0.100000 . . .	0.100000 . . .
Iteration 2	0.110000 . . .	0.010000 . . .
Iteration 3	0.011000 . . .	0.101000 . . .
Iteration 4	0.101100 . . .	0.010100 . . .
Iteration 5	0.010110 . . .	0.101010 . . .

Note that $1/2 = 0.1000$ is reached after one iteration. In fact $x_0 = 1$ and $x_0 = 0$ are the only starting positions that lead to $x = 1/2$.

The interested reader may be interested in investigating similar results for ternary and higher n-ary expansions with initial water amounts in John's glass of $\frac{1}{3}$, or, in general, $\frac{1}{n}$.

Further reading

Anyone wishing to read a further exposition of problems involving the method of cobwebs are referred to the books *A First Course in Chaotic Dynamical Systems* by Devaney (Chapters 4 and 5) [1] and *Understanding Nonlinear Dynamics* by Kaplan and Glass [2]. Students may also be interested in reading the article "Clarifying Compositions with Cobwebs" by Neger and Frame [3].

References

[1] R. L. Devaney, *A First Course in Chaotic Dynamical Systems: Theory and Experiment*, Addison-Wesley, 1992.

[2] D. Kaplan and L. Glass, *Understanding Nonlinear Dynamics*, Springer-Verlag, 1995.

[3] N. NEGER and M. FRAME, Clarifying Compositions with Cobwebs, College Mathematics Journal **34**, (2003) 196-204.